Many-Valued Logic

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1 Motivation

In *On Interpretation*, Aristotle discusses (inconclusively) the problem of future contingents. His example was the sentence, “There will be a sea-fight tomorrow.”, whose truth-value seems not to be fixed now. We can say that the states of affairs reported by such sentences are underdetermined by present circumstances.

One way of handling underdetermining sentences (it is a matter of dispute as to whether it was Aristotle’s way) is to represent the underdetermination by a special kind of truth value $I$, which can be read, “not-yet determined” or “underdetermined”. Such an approach is called three-valued logic. It is a logic in which $I$ is the third truth value.

Aside from underdetermination, another motivation for many-valued logic is indeterminacy or vagueness. The mark of a vague predicate is that it admits of satisfaction by degrees. Imagine that you are walking through the door from the living room into the dining room. At some stage, your left arm and left leg are still in the living room and your right arm and leg are in the dining room. Where are you? Dialetheic logicians (of whom more later in the course) answer that

- You are both in and not in each room.

Many-valued logicians replay that

- You neither are in either room nor are not in either room.

Given the truly massive numbers of vague predicates in any natural language, it could be argued that developing a solid many-valued logic is indispensable to a satisfactory semantics of any natural language.

2 Many-Valued Logics

In considering many-valued systems of logic, it is nearly automatic to begin with Łukasiewicz’s three valued system $\mathcal{L}^3$.\(^1\) The semantics for $\mathcal{L}^3$ include three values, T for truth, F, for

falsehood $I$, for some suitably interpreted third alternative. (It is in Łukasiewicz possibility; in Bočvar’s system it is interpreted as meaninglessness, and it is undefinedness in Kleene’s system.)

In $L^3$ it is not the case that all valuations $v_i$ are such that $v_i(\neg(\neg A \land \neg A) \land) = T$ or $v_i(\neg(\neg A \lor \neg A) \lor) = T$. That is, the classical laws of Non-Contradiction and Excluded Middle fail to hold.

We have $L^3$ the following valuation-equations ($L^3 = K_3$ in our textbook):

$L^3(\neg)$: \[
\neg(T) = F; \\
\neg(F) = T; \\
\neg(I) = I
\]

$L^3(\land)$: \[
\land(T, T) = T; \\
\land(T, I) = I; \\
\land(T, F) = F; \\
\land(I, T) = I; \\
\land(I, I) = I; \\
\land(I, F) = F; \\
\land(F, T) = F; \\
\land(F, I) = F; \\
\land(F, F) = F;
\]

$L^3(\lor)$: \[
\lor(T, T) = T; \\
\lor(T, I) = T; \\
\lor(T, F) = I; \\
\lor(I, T) = T; \\
\lor(I, I) = I; \\
\lor(I, F) = F; \\
\lor(F, I) = I; \\
\lor(F, F) = F;
\]

$L^3(\Rightarrow)$: \[
\Rightarrow(T, T) = T; \\
\Rightarrow(T, I) = I; \\
\Rightarrow(T, F) = F; \\
\Rightarrow(I, T) = T; \\
\Rightarrow(I, I) = I; \\
\Rightarrow(I, F) = F; \\
\Rightarrow(F, I) = T;
\]

The rules $L^3(\neg), ..., L^3(\Rightarrow)$ establish that, if $I$ is taken to be intermediate in ‘truthfulness’ between truth and falsehood, then the negation $\neg A^\gamma$ of a statement $A$ will always take as value the exact opposite of the value of $A$, except where $A$ is $I$ (in that event $\neg A^\gamma$ takes the same value as $A$); that a conjunction $B$ has the least-true value of its constituents; and that an implication $\Gamma(A) \Rightarrow (A)^\gamma$ is evaluated exactly as in classical systems (i.e., $v(\Gamma A \Rightarrow A^\gamma) = v(\Gamma A \lor A^\gamma)$) save that where $v(A) = v(A') = I$, $v(\Gamma A \Rightarrow A^\gamma)$ is adjusted to come out $T$ so as to preserve the validity of $\Gamma B \Rightarrow B^\gamma$. (We also remark that $v(\Gamma A \Leftarrow A^\gamma) = v(\Gamma A \Leftarrow A^\gamma) = v(\Gamma A \Rightarrow A^\gamma) = v(\Gamma A \Leftarrow A^\gamma)$, as in the usual two-valued case.)

Now a three- or $n$-valued logic is interesting only insofar as it differs structurally from its two-valued counterpart. For example if $\sigma$ is a sentence-connective of a many valued system $S$ and $\sigma'$ is its two valued counterpart, then the semantics of $S$ for $\sigma$ are standard if $S$ contains at least one designated value (e.g., $T$) and one contra-designated value (e.g., $F$), and $S$ provides that $v(\sigma(A)) = v(\sigma'(A)^\gamma)$ if $v(\sigma(A)^\gamma) = T$ or $v(\sigma(A)^\gamma) = F$, for all $\sigma$ and $\sigma'$ having identical matrices for $T$ and $F$. $L^3$ is in this sense standard.

A semantics $S$ for an $n$-valued negation-connective is Kleene-regular iff $v(\neg(A)^\gamma) = T$ or $F$ just in case $v(A) = F$ or $T$.

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5However, the systems of Post are not standard. See E.L. Post, “Introduction to the General Theory of Elementary Propositions”, American Journal of Mathematics, 43, 163–185.
The negation modality is stable iff nowhere do the semantics for ‘¬’ permit a valuation v such that v(A) = v(⌜¬A⌝).

\(L^3\) is Kleene-regular but unstable with respect to negation. It would be interesting if it were to turn out either (1) that for no sentence A and for no valuation \(v_i\) is it the case that \(v_i(⌜A \land \neg A⌝) = T\); or (2) that for no sentences A and B is it the case that \(v_i(⌜(A \lor B) \land \neg B⌝) = T\), for all valuations \(v_i\) (i.e., Disjunctive Syllologism does not obtain). It is easily verified that all three conditions are met in \(L^3\).

However, there is, in \(L^3\), a deep disadvantage according to some logicians. As we have seen, negation in \(L^3\) is not stable. It is Kleene-regular, of course, since in \(L^3\) \(\neg(T) = F\) and \(\neg(F) = T\). But it is not stable because it allows for self-annulment: I = \(\neg(I)\).

3 The System \(L^4\)

Suppose, then, that we explore a 4-valued sentence logic, \(L^4\), with a value, T, for truth, F, for falsehood, I₁ for truth-likeness and I₂ for falsity-likeness. We designate T and I₁, and, for the present, contra-designate F.

The connectives ‘¬’, ‘∧’ and ‘⊃’ are defined as follows:

<table>
<thead>
<tr>
<th>Negation</th>
<th>Conjunction</th>
<th>Implication⁶</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (\neg\neg A)</td>
<td>A B (A \land B)</td>
<td>(A \supset B)</td>
</tr>
<tr>
<td>T F</td>
<td>T T T</td>
<td>T</td>
</tr>
<tr>
<td>I₁ I₂</td>
<td>I₂ T I₂</td>
<td>T</td>
</tr>
<tr>
<td>I₂ I₁</td>
<td>F T F</td>
<td>T</td>
</tr>
<tr>
<td>F T</td>
<td>T F F</td>
<td>F</td>
</tr>
<tr>
<td>I₁ F F</td>
<td>F I₁</td>
<td>I₁</td>
</tr>
<tr>
<td>I₂ F F</td>
<td>F I₂</td>
<td>I₁</td>
</tr>
<tr>
<td>F F F</td>
<td>F I₁</td>
<td>T</td>
</tr>
<tr>
<td>T I₁ I₁</td>
<td>I₁ I₁</td>
<td>I₁</td>
</tr>
<tr>
<td>I₁ I₂ I₁</td>
<td>I₂ I₁</td>
<td>I₁</td>
</tr>
<tr>
<td>I₂ I₁ I₂</td>
<td>T I₂</td>
<td>I₂</td>
</tr>
<tr>
<td>I₁ I₂ I₂</td>
<td>F I₁</td>
<td>I₂</td>
</tr>
<tr>
<td>I₂ I₂ I₂</td>
<td>I₂ I₂</td>
<td>I₁</td>
</tr>
<tr>
<td>I₂ I₁ I₂</td>
<td>F I₂</td>
<td>I₂</td>
</tr>
<tr>
<td>F I₂ F</td>
<td>F I₁</td>
<td>T</td>
</tr>
</tbody>
</table>

⁶The connectives in \(L^4\) do not all interdefine. For example, \(A \supset B\) and \(\neg(\neg A \land \neg B)\) do not exactly agree in truth-value; the implication takes I₁ in the sixth row, whereas the negated conjunction takes I₂. True, it seems slightly counterintuitive to evaluate \(A \supset B\) as I₁, when its antecedent is ‘truer’ than its consequent. But, if we altered the defining matrix of ‘⊃’ so as to change the value of \(A \supset B\) at row six to I₂, \(L^4\) would be strictly isomorphic to classical sentence logic, provided T and I₁ were designated and F and I₂ were counter-designated. So we shall, for the present, tolerate row six in the matrix for
Now the negation-function of $L^4$ is both Kleene-regular and stable, and so is non-
self-annulling.

As in $L^3$, neither the Law of Non-Contradiction nor the Law of Excluded Middle is
classically valid in $L^4$. And here too, as in $L^3$, neither $\neg(A \land \neg A) \supset B^\top$ nor $\neg((A \lor B) \land \neg A) \supset B^\top$ is classically valid.

However these selfsame conditionals so behave in $L^4$ that for all valuations, $v_1,
\vdash v_i(\neg((A \lor B) \land \neg B)^\top) = T(\text{or}_I)$, and $v_i(\neg((A \lor B) \land \neg B)^\top) = I(\text{or}_I)$; that is, these
conditional are non-classically valid in the sense that under every valuation they are
given one or other designated value.

### 4 The Composition System $L^8$

$L^8$ is an 8-valued composition of the classical sublogics $F_1, F_2, \text{ and } R$. That is to say,
$L^8$ is understood to be the Cartesian product, $F_1 \times F_2 \times R$, in the precise sense that
every one of the eight truth values of $L^8$ is a member of the Cartesian product of one
or other of the truth-values of $F_1 \times F_2 \times R$. Since $F_1, F_2, \text{ and } R$ are presumed to be
classically truth-valued, and putting ‘$T$’ for truth and ‘$F$’ for falsehood, the values of $L^8$
are ordered triples of Ts and Fs. In particular, if $A$ is a sentence of $L^8$ and $j(1 \leq j \leq 8)$
is its truth-value then $j = \langle \pm T, \pm T, \pm T \rangle$ where $A$ receives the classical truth-value $\pm T$
in $F_1, \pm T$ in $F_2$, and $\pm T$ in $R$, and $+T = T$ and $-T = F$.

The truth-values of $L^8$ are

\[
\begin{array}{c}
1 = \langle TTT \rangle \\
2 = \langle TTF \rangle \\
3 = \langle TFT \rangle \\
4 = \langle TFF \rangle \\
5 = \langle FTT \rangle \\
6 = \langle FTF \rangle \\
7 = \langle FFT \rangle \\
8 = \langle FFF \rangle \\
\end{array}
\]

Negation is given by the matrix:

\[
\begin{array}{c|c}
A & \neg A^\top \\
\hline
1 & 8 \\
2 & 7 \\
3 & 6 \\
4 & 5 \\
5 & 4 \\
6 & 3 \\
7 & 2 \\
8 & 1 \\
\end{array}
\]

‘$\top$’, as is.
Conjunction is defined by the following abbreviating equations.

1. \(1 \land 1 = 1; 1 \land 2 = 2; 1 \land 3 = 3; 1 \land 4 = 4; 1 \land 5 = 5; 1 \land 6 = 6; 1 \land 7 = 7; 1 \land 8 = 8\)
2. \(2 \land 1 = 2; 2 \land 2 = 2; 2 \land 3 = 4; 2 \land 4 = 4; 2 \land 5 = 6; 2 \land 6 = 8; 2 \land 7 = 8; 2 \land 8 = 8\)
3. \(3 \land 1 = 3; 3 \land 2 = 4; 3 \land 3 = 3; 3 \land 4 = 4; 3 \land 5 = 7; 3 \land 6 = 8; 3 \land 7 = 7; 3 \land 8 = 8\)
4. \(4 \land 1 = 4; 4 \land 2 = 4; 4 \land 3 = 4; 4 \land 4 = 4; 4 \land 5 = 8; 4 \land 6 = 8; 4 \land 7 = 8; 4 \land 8 = 8\)
5. \(5 \land 1 = 5; 5 \land 2 = 6; 5 \land 3 = 7; 5 \land 4 = 8; 5 \land 5 = 5; 5 \land 6 = 6; 5 \land 7 = 6; 5 \land 8 = 8\)
6. \(6 \land 1 = 6; 6 \land 2 = 8; 6 \land 3 = 8; 6 \land 4 = 8; 6 \land 5 = 6; 6 \land 6 = 6; 6 \land 7 = 8; 6 \land 8 = 8\)
7. \(7 \land 1 = 7; 7 \land 2 = 8; 7 \land 3 = 7; 7 \land 4 = 8; 7 \land 5 = 6; 7 \land 6 = 8; 7 \land 7 = 7; 7 \land 8 = 8\)
8. \(8 \land 1 = 8; 8 \land 2 = 8; 8 \land 3 = 8; 8 \land 4 = 8; 8 \land 5 = 8; 8 \land 6 = 8; 8 \land 7 = 8; 8 \land 8 = 8\)

Note, in particular, that “\(1 \land 1 = 1\)” is short for “if \(v(A) = 1\) and \(v(B) = 1\), then \(v(\llbracket A \land B \rrbracket) = 1\).”

The values for ‘\(\land\)’ are computed as follows:

\[i \land h = \langle \pm T, \pm T, \pm T \rangle \land \langle \pm T, \pm T, \pm T \rangle = \langle \pm T \land \pm T, \pm T \land \pm T, \pm T \land \pm T \rangle = \langle \pm T, \pm T, \pm T \rangle,\]

according as \(\pm T\) is \(T\) or \(F\), and the following conditions are met:

\[T \land T = T, T \land F = F, F \land T = F, F \land F = F.\]

As an example, suppose we have the formula \(\llbracket A \land B \rrbracket\), and that \(A\) receives the value of 4 and \(B\) the value of 3. To compute the value of \(\llbracket A \land B \rrbracket\) we write, 4 \(\land\) 3.

Then replacing 4 and 3 with the triple of classical values with which they have been identified, we have,

\[\langle TFF \rangle \land \langle TTT \rangle.\]

But since in general \(\langle \pm T, \pm T, \ldots, \pm T \rangle \land \langle \pm T, \pm T, \ldots, \pm T \rangle = \langle \pm T \land \pm T, \ldots, \pm T \land \pm T \rangle\),
we write \(\langle T \land T, F \land F, F \land T \rangle\),
which reduces to

\[\langle T, F, F \rangle.\]

And, since this triple is identified with the value 4, we conclude that, 4 is the truth value of \(\llbracket A \land B \rrbracket\) if 4 is the truth-value of \(A\) and 3 of \(B\). And this is just what the third entry from the left of row (4) says.

In \(L^8\), the values 1,2,3 and 4 are designated, and the values 5,6,7 and 8 are contradedignated. The principle by which it is determined whether an \(L^8\)-value is designated is by whether or not its first element is classically designated.
It is easy to see that negation in $L^8$ is stable; nowhere do we have $v(A) = v(\neg \neg A)$. And if we were to identify the $L^8$-value $1 = (TTT)$ with the classical value $T$, and $8 = (FFF)$ with $F$, negation in $L^8$ would be Kleene-regular, as well. In fact, since negation in $L^8$ is mirror-imagistic, it is an orthodox notion of negation; for it always sends a designated value into a contradesignated value, and vice versa.

An $L^8$-sentence $A$ is $L^8$-valid iff it receives a designated value for all $L^8$-assignments to its components. An $L^8$-formula is $L^8$-inconsistent if it receives none but contra-designated values. $A$ is $L^8$-valid iff $\neg \neg A$ is $L^8$-inconsistent.

Some valid $L^8$-sentences are classically valid, putting $1 = T$ and $8 = F$. Examples include Modus Ponens, Non-Contradiction, Double Negation and Excluded Middle. Bivalence, of course, does not hold; however, $L^8$ has a ‘bivalent climate’ about it, in the sense that its every sentence is either designated or not.

Composition logics have been investigated, for example by Prior, with a view to providing a formal semantics for chronological languages, and by Rose, in an attempt to find a common semantics for different systems of geometry. One of their principle advantages is that these are semantics that need involve objects no more arcane than the ordinary truth values and $n$-tuples of these.

5

Not all logicians are, to say the least, sold on the many-valued approach. Here is Dana Scott on the matter (“Advice on Modal Logic”, in Karel Lambert, editor, Philosophical Problems in Logic, Amsterdam: North-Holland 1970, 143–174; p. 153):

Yes, yes I can hear the objections being shouted from all corners.

If one is going to use undefined terms why not undefined truth values? Is not that more natural? Maybe so, but I have yet to see a really workable [many-] valued logic. I know it can be defined, and at least four times a year someone comes up with the idea anew, but it has not really been developed to the point where one could say it is pleasant to work with.

It is doubtless true, as Quine has said, that

Primarily the motivation of [many-valued] studies has been abstractedly mathematical; the pursuit of analogy and generalization. Studied in this spirit, many-valued logic is logic only analogically speaking; it is uninterpreted theory, abstract algebra (The Philosophy of Logic, p. 84).

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9See also Rescher, Topics in Philosophical Logic, 89–90.
10This, the leading idea of many-valued composition logics, must be credited to Post in “Introduction to the General Theory of Elementary Propositions”.
11Scott’s complaint specifies three-valued systems.
Yet, think (commands Quine) of “the handicap of having to think within a deviant logic. The price is perhaps not quite prohibitive, but the returns had better be good” (The Philosophy of Logic, p. 86).

6 A Further Motivation

Consider an example made famous by Russell:

- The present King of France is bald.

It is apparent that if true, there is presently a king who rules France. And we would seem to have it that if false, then the present king of France isn’t bald. But wouldn’t this also mean that there is a present king of France?

One way around the problem of sentences that contain non-denoting (or empty) terms is to embed them in a three-valued logic in which such sentences are assigned the third value.

However, Russell’s own solution avoids the many-valued option.