

Let us consider now a compactly supported measurable bounded function $\lambda(\mathbf{x})$ satisfying $\int \lambda(\mathbf{x}) d\mathbf{x} = 1$ and

$$\iint \beta(\mathbf{x})\lambda(\mathbf{y})(\mathbf{x}-\mathbf{y})^{\mathbf{k}} d\mathbf{x} d\mathbf{y} = \begin{cases} 1, & \text{for } \mathbf{k} = 0, \\ 0, & \text{for } 0 < |\mathbf{k}| \leq m, \end{cases}$$

where $\mathbf{k} = (k_1, \dots, k_d)$, $|\mathbf{k}| = k_1 + \dots + k_d$, and $(\mathbf{x}-\mathbf{y})^{\mathbf{k}} = (x_1 - y_1)^{k_1} \dots (x_d - y_d)^{k_d}$.

Then we define a restriction operator r_h : for $f \in L_{loc}^1(\mathbb{R}^d)$, we denote $f_h^j = \frac{1}{h^d} \int \lambda(\frac{\mathbf{x}}{h} - j) f(\mathbf{x}) d\mathbf{x}$, and define $r_h f = \sum_j f_h^j \chi(\frac{\mathbf{x}}{h} - j)$.

We have the well-known estimates (where c denotes different constants which do not depend on h):

- (1) If $f_h \in L^2(\mathbb{R}^d)$, we have $\|r_h f\|_{L^2} \leq c \|f\|_{L^2}$.
- (2) If $f_h \in L^2(\mathbb{R}^d)$, we have $p_h f_h \in H^m$ and $\|p_h f_h\|_{H^m} \leq \frac{c}{h^m} \|f_h\|_{L^2}$, moreover $c \|f_h\|_{L^2} \leq \|p_h f_h\|_{L^2} \leq \|f_h\|_{L^2}$, where $c > 0$ does not depend on h .

It follows that p_h is an isomorphism from the space of the functions f_h which are square integrable onto a subspace F_h of H^m .

- (3) If $f \in H^{m+1}(\mathbb{R}^d)$, for $0 \leq k \leq s \leq m+1$ and $k \leq m$, we have:

$$\|f - p_h r_h f\|_{H^k} \leq c h^{s-k} \|f\|_{H^s}$$

The periodic case. Let us suppose that $h = 1/N$. We denote H_{per}^m the Sobolev space H^m on the d -dimensional torus $(\mathbb{R}/\mathbb{Z})^d$, and $F_h(Q_d)$ the space of the restrictions to Q_d of the functions of the form $\sum_j f_h^j \beta(\frac{\mathbf{x}}{h} - j)$ which are \mathbb{Z}^d -periodic (i.e., $f_h^j = f_h^{j+N_i}$, for all j, i in \mathbb{Z}^d). $F_h(Q_d)$ is endowed with the L^2 scalar product.

Obviously, if f and f_h are \mathbb{Z}^d -periodic, so are $r_h f$ and $p_h f_h$. And the following estimates hold:

- (1') If $f \in L_{per}^2$, we have $\|r_h f\|_{L^2(Q_d)} \leq c \|f\|_{L^2(Q_d)}$.
- (2') If f is \mathbb{Z}^d -periodic, we have $\|p_h f_h\|_{H_{per}^m} \leq \frac{c}{h^m} \|f_h\|_{L^2(Q_d)}$, and

$$c \|f_h\|_{L^2(Q_d)} \leq \|p_h f_h\|_{L^2(Q_d)} \leq \|f_h\|_{L^2(Q_d)}$$

- (3') If $f \in H_{per}^{m+1}$, for $0 \leq k \leq s \leq m+1$ and $k \leq m$, we have

$$\|f - p_h r_h f\|_{H_{per}^k} \leq c h^{s-k} \|f\|_{H_{per}^s}$$

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