Please memorize the axioms and rules for S1, S2, S4 and S5.


**Rules**

*Substitution.* If A’ is exactly like A except for containing a wff C at some places where A contains B then ⊢ (B ≡ C) ⊃ (A’ ≡ A). In words, if A’ and A are the same wff except for different but strictly equivalent parts, then A’ is strictly equivalent to A.

*Adjunction.* If ⊢ A and ⊢ B then ⊢ (A ∧ B).

*Inference.* If ⊢ A and ⊢ (A → B) then ⊢ B.

*Definitions:* □A iff ~◊ ~A; ◊A iff ~□ ~A; A → B iff ~◊(A ∧ ~B)

**Axioms for S1**

B1. (A ∧ B) → (B ∧ A)  
B2. (A ∧ B) → B  
B3. A → (A ∧ A)  
B4. ((A ∧ B) ∧ C) → (A ∧ (B ∧ C))  
B5. (A → B) → (B → C) → (A → (B ∧ C))  
B6. ((A → B) ∧ (B → C)) → (A → C)  
B7. (A ∧ (A → B)) → B


**Axioms for S2**

B1-B7 + B8: ◊(A ∧ B) → ◊A.

**Axioms for S3**
B1-B7 + A8: (A → B) → (¬B → ¬A).

Axioms for S4

B1-B7 + C10: ¬¬A → ¬¬¬¬A.

Axioms for S5

Axioms of S2 + C11: ◊A → ¬◊¬A.


Axioms of S2 + C13: ◊◊A.


Axioms for S3 + C13: ◊◊A.

Axioms for S8 (see above)

Axioms for S3 + ¬◊¬◊A.

2. The Gödel Systems, 1933 (Kurt Gödel, “Eine Interpretation des intuitionistischen Aussagenkalküls”, Ergebnisse eines mathematischen Kolloquiums Heft 4, 1933, 39-40)).

Rule

RL: If ⊢ A then ⊢ □A

Axioms for Gödel’s Basic System

A.1: □A ⊢ A.
A.2: □(A ⊢ B) ⊢ (□A ⊢ □B).

Axioms for Gödel’s Original System

A.1-A.2 + A.4: □A ⊢ □□A
(Notes: 1. Gödel’s Original System is equivalent to S4.
   2. Gödel’s Basic System when supplemented by the axiom A.5 (◊A ⊢ □◊A) is equivalent to S5.
   3. When Gödel’s Basic System is supplemented by “Brouwer’s” axiom
A.3 (A ⊳ □ ◊A), the result is equivalent to Brouwer’s System.


**Rule**

Feys’ Rule 25.2 is the same as Gödel’s RL.

**Axioms**

Feys’ axiom 25.3 is Gödel’s A.2
Feys’ axiom 23.11 is A ⊳ ◊A.

*Note:* Feys’s System is equivalent to Gödel’s Basic System.


**Rules**

The rules of a system of classical propositional logic plus:

*Extensionality:* If ⊢ A ≡ B then ⊢ ◊A ≡ ◊B.
*Tautology:* If ⊢ A then ⊢ □A.

**Axioms for M**

The axiom of possibility: A ⊳ ◊A
The axiom of distribution: ◊(A ∨ B) ≡ (◊A ∨ ◊B).

**Axioms for M’**

The axioms for M plus:
The first axiom of reduction: ◊◊A ⊳ ◊A.

**Axioms for M’’**

The axioms for M plus:
The second axiom of reduction: ◊ ~◊A ⊳ ~◊A.

5. Interrelations Between the Systems

Gödel’s Basic System
Feys’ System
1. $\rightarrow$ expresses containment.

2. Systems above line A have rule RL (if $\vdash A$ then $\vdash \square A$)

3. Systems below line A don’t have RL.

4. Systems below line B have $\vdash \Diamond \Diamond A$.

5. Systems above line B don’t have $\vdash \Diamond \Diamond A$.

6. Systems above line A are incompatible with $\Diamond A$.

7. Systems below line B are incompatible with RL.

On the last page of this note is a more recent and comprehensive chart. It is taken with permission and my thanks from Andrew Irvine’s “S7”, Journal of Applied Logic, 11 (2013), 525. For the border key, see the paper.

*Further optional reading*